

# Generalized fermion algebra

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## Abstract

A one-parameter generalized fermion algebra  $\mathcal{B}_\kappa(1)$  is introduced. The Fock representation is studied. The associated coherent states are constructed and the polynomial representation, in the Bargmann sense, is derived. A special attention is devoted to the limiting case  $\kappa \rightarrow 0$  where the fermionic coherent states, labeled by Grassmann variables, are obtained. The physical relevance of the algebra is illustrated throughout Calogero-Sutherland system.

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# 1 Introduction and motivations

Over the last two decades, major effort has been directed towards the generalized Weyl-Heisenberg algebras. In connection with the theory of quantum algebras and in the spirit of the pioneering works [1, 2, 3] a wide class of non linear bosonic algebras were investigated from many perspectives and for different purposes. The common feature of all these generalizations is their structural similarity and subsequently they can be described in a unified mathematical framework [4]. Indeed, any generalized Weyl-Heisenberg algebra can be defined by the basic structure relations

$$a^+a^- = F_-(N) \quad a^-a^+ = F_-(N+1) \quad [N, a^\pm] = \pm a^\pm$$

where the structure function  $F_-(N)$  ( $N$  stands for the usual number operator) defined as the product of the creation and annihilation operators  $a^+$  and  $a^-$ . Equivalently, the structure relations of the extended Weyl-Heisenberg write

$$[a^+, a^-] = F_-(N+1) - F_-(N) \equiv G_-(N) \quad [N, a^\pm] = \pm a^\pm.$$

The function  $F(N)$  characterizes the generalization or deformation scheme. Several possible generalizations can be defined, each one might be useful for different purposes (see for instance [5, 6, 7, 8, 9]). For instance, the polynomial Weyl-Heisenberg algebras, corresponding to the case where the structure function  $F_-(N)$  is a polynomial in  $N$ , are of special interest for quantum systems with non-linear discrete spectrum [10, 11, 12, 13]. In this context, an interesting situation corresponds to the case where  $G_-(N)$  is linear in  $N$  :

$$G_-(N) = 1 + 2\kappa N$$

The usual harmonic oscillator is recovered when  $\kappa = 0$ . This algebra includes  $su(2)$  for  $\kappa < 0$ ,  $su(1, 1)$  for  $\kappa > 0$  and it related to some exactly solvable potentials like Morse and Pöschl-Teller ones [11].

In other hand, paralleling the extension or deformation of standard structure relations defining the bosonic oscillator algebra, several extensions schemes of the fermionic algebra were studied in [14]. In particular, the authors concluded that all deformed fermionic algebras, in the context of the  $q$ -deformations inherited from quantum Lie algebras, are isomorphically equivalent to the non-deformed fermionic algebra when the deformation parameter  $q$  is generic. For  $q$  a root of unity, deformation schemes of fermions provide a nice mathematical framework to define objects interpolating between bosons and fermions (see [15] and references therein). In this respect, motivated by the polynomial extension of the usual bosonic algebra presented briefly above, we shall mainly focus on the generalization of the usual fermion algebra given by the following basic structure relations

$$f^+f^- = F_+(N) \quad f^-f^+ = F_+(N+1) \quad [N, f^\pm] = \pm f^\pm$$

which can be re-equated also as

$$\{f^+, f^-\} = F_+(N+1) + F_+(N) \equiv G_+(N) \quad [N, f^\pm] = \pm f^\pm.$$

In Section 2, we consider a particular, but nevertheless very large class of extended fermion algebra. We shall pay attention to the one parameter algebra where the function  $G_+(N)$  is linear in the number operator  $N$ . The Fock representation space is explicitly defined. Section 3 deals with the connection

with this particular class of generalized fermion algebra and the two-body Calogero-Sutherland model. We also discuss the bosonization procedure, leading to the generalized algebra  $\mathcal{A}_\kappa(1)$  introduced in [11], by using a suitable boson mapping in terms of generalized fermion generators. In Section 4, an analytical representation of the algebra is derived. It is based on the associated coherent states. Finally, normal ordering process is discussed in Section 5. The expressions of the analogue of Strling and Bell operators are obtained. Concluding remarks and possible prolongations of the present work close this paper.

## 2 Generalized fermion algebra

As mentioned above, this work focuses on the special class of the generalized or extended fermion algebra where  $G_+(N)$  is linear in the number operator. Analogously to the generalized boson or Weyl-Heisenberg algebra  $\mathcal{A}_\kappa(1)$  discussed in [11], it will be denoted by  $\mathcal{B}_\kappa(1)$ . In this case, the generators  $\{f^+, f^-, N\}$ , satisfy the structure relations

$$\{f^-, f^+\} = \mathbb{I} + 2\kappa N \quad [N, f^-] = -f^-, \quad [N, f^+] = +f^+, \quad (1)$$

where  $\mathbb{I}$  is the unity operator and  $\kappa$  is a real parameter characterizing the deviation from the usual fermion algebra which is recovered for  $\kappa = 0$ . It is interesting to note that the generalized fermionic operators obey the following identities

$$f^-(f^+)^n = (-1)^n (f^+)^n f^- + \left\{ \frac{1 - (-1)^n}{2} (1 + \kappa + 2\kappa N) + (-1)^n \kappa n \right\} (f^+)^{n-1} \quad (2)$$

and

$$(f^-)^n f^+ = (-1)^n f^+ (f^-)^n + (f^-)^{n-1} \left\{ \frac{1 - (-1)^n}{2} (1 + \kappa + 2\kappa N) + (-1)^n \kappa n \right\}. \quad (3)$$

From these equations, one can see that in the specific case  $\kappa = 0$ , the operators  $(f^+)^2$  and  $(f^-)^2$  belong to the center of the algebra. They satisfy the nilpotency relation  $(f^+)^2 = (f^-)^2 = 0$  that reflects the Pauli exclusion principle. Except this particular situation, there not exists any integer  $n$  such that the ladder operators  $f^+$  and  $f^-$  are nilpotent. Consequently, for  $\kappa \neq 0$ , the representations of generalized fermion are infinite dimensional. The Fock representation of this algebra is given by means of a complete set of orthonormal states  $\mathcal{F} = \{|n\rangle, n \in \mathbb{N}\}$  which are eigenstates of the number operator  $N$ ,  $N|n\rangle = n|n\rangle$ . In this representation, the vacuum state is defined as  $f^-|0\rangle = 0$  and the ortho-normalized eigenstates are constructed by successive applications of the creation operator  $f^+$ . We define the actions of creation and annihilation operators as

$$f^-|n\rangle = \sqrt{F_+(n)}|n-1\rangle, \quad f^+|n\rangle = \sqrt{F_+(n+1)}|n+1\rangle \quad (4)$$

where the structure function  $F_+(\cdot)$  is an analytic function satisfying the two following conditions

$$F_+(0) = 0 \quad \text{and} \quad F_+(n) > 0, \quad n = 1, \dots \quad (5)$$

The action of the operator  $G_+(N)$  is given by

$$G_+(N)|n\rangle = (1 + 2\kappa n)|n\rangle$$

It is simple to check that, in the Fock space  $\mathcal{F}$ , the operators  $f^+$  and  $f^-$  are mutually adjoint,  $f^+ = (f^-)^\dagger$ . It is also easy to verify that the structure function  $F_+(\cdot)$  satisfies the following recursion formula

$$F_+(n+1) + F_+(n) = G_+(n), \quad (6)$$

where  $G_+(n) = 1 + 2\kappa n$ . By simple iteration, we get

$$F_+(n) = (-)^{n-1} \sum_{m=0}^{n-1} (-)^m G_+(m), \quad (7)$$

and the explicit form of the structure function  $F_+(N)$  is given by

$$F_+(N) = \frac{1}{2}(1 - (-1)^N)(1 + \kappa(N-1)) + \frac{\kappa}{2}N(1 + (-1)^N). \quad (8)$$

It follows that the concrete form of the actions of the creation and annihilation operators (4) are

$$f^-|n\rangle = \sqrt{\frac{1}{2}(1 - (-1)^n)(1 + \kappa(n-1)) + \frac{\kappa}{2}n(1 + (-1)^n)}|n-1\rangle \quad (9)$$

$$f^+|n\rangle = \sqrt{\frac{1}{2}(1 + (-1)^n)(1 + \kappa n) + \frac{\kappa}{2}(n+1)(1 - (-1)^n)}|n+1\rangle. \quad (10)$$

From Eq.(8), the structure function is given by

$$F_+(n) = \kappa n$$

for  $n$  even and for  $n$  odd it writes

$$F_+(n) = 1 + \kappa(n-1).$$

It is important to stress that  $\kappa$  should be positive to ensure the positivity of  $F_+(n)$ . Obviously, the representation space of the algebra  $\mathcal{B}_\kappa(1)$  is infinite dimensional except the limiting case  $\kappa = 0$  where the usual fermion algebra is recovered. In this case, the Fock space  $\mathcal{F}$  is two dimensional and comprises only the states  $|0\rangle$  and  $|1\rangle$ .

### 3 The Hamiltonian and two-body Calogero-Sutherland spectrum

The algebra  $\mathcal{B}_\kappa(1)$  is interesting from two different aspects. The first concerns its relevance for the study of some one-dimensional exactly solvable potentials. The second is related to its connection the  $\mathbb{Z}_k$  graded Weyl-Heisenberg algebra  $W_k$  introduced in [10] and the generalized Weyl-Heisenberg algebra  $\mathcal{A}_\kappa(1)$  defined in [11].

Concerning the first aspect, the algebra  $\mathcal{B}_\kappa(1)$  can be related to the two-body Calogero-Sutherland model. For this end, we notice that the structure function  $F_+(N) \equiv f^+ f^-$  can be also written as

$$F_+(N) = \kappa N + (1 - \kappa)\Pi_1 \quad (11)$$

where the operator  $\Pi_1$  projects on the odd number Fock states. It is defined by

$$\Pi_1 = \frac{1 - (-)^N}{2}$$

which is orthogonal to the operator

$$\Pi_0 = \frac{1 + (-)^N}{2}$$

projecting on even Fock states. This induces a  $\mathbb{Z}_2$  graduation of the Fock space. Accordingly, the products  $f^+ f^-$  and  $f^- f^+$  are

$$f^+ f^- = \kappa N + (1 - \kappa)\Pi_1 \quad f^- f^+ = \kappa(N + 1) + (1 - \kappa)\Pi_0$$

and the commutator between the operators  $f^+$  and  $f^-$  rewrites

$$[f^-, f^+] = \Pi_0 + (2\kappa - 1)\Pi_1 \quad (12)$$

which is exactly the  $\mathbb{Z}_2$  graded Weyl-Heisenberg algebra  $W_2$  [10]. In view of this  $\mathbb{Z}_2$  graduation, the algebra  $\mathcal{B}_\kappa(1)$  is relevant for the description of the energy spectra of the one dimensional two-body Calogero-Sutherland model. Indeed, the spectrum of the operator  $f^+ f^-$  given by

$$f^+ f^- |2n\rangle = 2\kappa n |2n\rangle \quad f^+ f^- |2n + 1\rangle = (2\kappa n + 1) |2n + 1\rangle$$

coincides with the Hamiltonian corresponding to the Calogero-Sutherland potential (in  $x$ -representation)

$$V_0(x, \kappa) = \frac{\kappa^2}{4}x^2 + \frac{1 - \kappa^2}{4\kappa^2} \frac{1}{x^2} - \left(\kappa - \frac{1}{2}\right).$$

This agrees with the results of the Ref. [10]. Furthermore, the spectrum of the operator  $f^- f^+$ :

$$f^- f^+ |2n\rangle = (2\kappa n + 1) |2n\rangle \quad f^- f^+ |2n + 1\rangle = (2\kappa n + 2) |2n + 1\rangle$$

coincides with the Hamiltonian of a two body system embedded in the Calogero-Sutherland of the form

$$V_1(x, \kappa) = \frac{\kappa^2}{4}x^2 - \frac{(1 - \kappa)(3\kappa - 1)}{4\kappa^2} \frac{1}{x^2} + \frac{1}{2}.$$

Using the super-symmetric quantum mechanics techniques, it is simply verified that  $V_1(x, \kappa)$  is the super-symmetric partner of  $V_0(x, \kappa)$  in accordance with the fact that the operators  $f^+ f^-$  and  $f^- f^+$  are isospectral. The first operator has the following sequence of eigenvalues  $0, 1, 2\kappa, 2\kappa + 1, 4\kappa, 4\kappa + 1, \dots$ , and for  $f^- f^+$  we have  $1, 2\kappa, 2\kappa + 1, 4\kappa, 4\kappa + 1, \dots$ . The energy levels are not equidistant and the energy gaps between two successive states are 1 or  $2\kappa - 1$ .

In other hand, the algebra  $\mathcal{B}_\kappa(1)$  can be related to the generalized Weyl-Heisenberg  $\mathcal{A}_\kappa(1)$  which has been shown useful in describing some exactly solvable Hamiltonians [10, 11]. For this purpose, we define the operators

$$X^+ = (f^+)^2 \quad X^- = (f^-)^2$$

in terms of the creation and annihilation operators  $f^+$  and  $f^-$ . Clearly, this realization is not possible with usual creation and annihilation fermionic operators ( $\kappa = 0$ ) satisfying the relations  $(f^+)^2 = 0$  and  $(f^-)^2 = 0$ . In view of the equation (9) and (10), we have

$$X^+ |n\rangle = \sqrt{F_+(n+2)F_+(n+1)} |n+2\rangle \quad X^- |n\rangle = \sqrt{F_+(n)F_+(n-1)} |n-2\rangle.$$

Then, one can write the operator products  $X^+ X^-$  and  $X^- X^+$  as follows

$$X^+ X^- = F_+(N)F_+(N-1) \equiv F(N) \quad X^- X^+ = F_+(N+2)F_+(N+1) \equiv F(N+2).$$

where the structure function  $F_+(N)$  is given by (8). The expression of the new structure function  $F(N)$  reads

$$F(N) = \kappa^2 N(N-1) + \kappa(\kappa-1)N - \kappa(\kappa-1) \frac{1 - (-)^N}{2},$$

and one has the following commutation relation

$$[X^-, X^+] = 2\kappa(2\kappa N + 1)$$

to be compared with one satisfied by the ladder operators of the generalized Weyl-Heisenberg algebra. In fact, setting

$$a^\pm = \frac{X^\pm}{\sqrt{2\kappa}},$$

one finds

$$[a^-, a^+] = 2\kappa N + 1 \quad [N, a^\pm] = \pm 2a^\pm. \quad (13)$$

corresponding to the structure relations defining the algebra  $A_\kappa(1)$ . This mapping to pass from the generalized fermion algebra to generalized boson algebra provides another example of the relevance of the algebra  $\mathcal{B}_\kappa(1)$  in the context of quantum mechanical systems possessing quadratic spectrum. In fact, the connection between the algebra defined by (13) and one-dimensional solvable potentials such as Morse or Pöschl-Teller potentials was discussed in [11]. Accordingly, the generalized fermion algebra  $\mathcal{B}_\kappa(1)$  provides the algebraic framework to deal with exactly solvable potentials with linear as well as quadratic spectra.

## 4 Coherent states and Bargmann representation

For the generalized fermion algebra  $\mathcal{B}_\kappa(1)$  a realization à la Bargmann can be found. This realization is the extended version of Bargmann representation for the usual fermions. In general, the derivation of these representations is the same as the usual fermion but the notion of the derivative and consequently the integral should be redefined. The appropriate way to approach the Bargmann representation for the algebra  $\mathcal{B}_\kappa(1)$  is the coherent states formalism. A coherent state for a generalized fermion is defined as the eigenvector of the annihilation operator

$$f^-|z\rangle = z|z\rangle \quad (14)$$

where  $z$  is a complex variable. We shall assume for now that  $\kappa \neq 0$ . The special case  $\kappa = 0$  where the Fock space has two dimensional basis will be discussed hereafter as a limiting case. To anticipate, we shall show that the bosonic variable  $z$  tends to a Grassmann variable to obtain the usual coherent states for an ordinary fermion. To solve the the eigenvalue equation (14), we expand the state  $|z\rangle$  in the Fock space basis  $|n\rangle$

$$|z\rangle = \sum_{n=0}^{\infty} c_{2n}(z)|2n\rangle + \sum_{n=0}^{\infty} c_{2n+1}(z)|2n+1\rangle. \quad (15)$$

Substituting this expression in (14), we get the following recurrence relation

$$z c_{2n}(z) = \sqrt{1 + 2\kappa n} c_{2n+1}(z) \quad z c_{2n+1}(z) = \sqrt{2\kappa(n+1)} c_{2n+2}(z) \quad (16)$$

which leads to

$$c_{2n}(z) = c_0(z) \frac{z^{2n}}{\sqrt{(2\kappa)^{2n} n! \Gamma(\frac{1}{2\kappa} + n)}} \quad c_{2n+1}(z) = c_0(z) \frac{z^{2n+1}}{\sqrt{(2\kappa)^{2n+1} n! \Gamma(\frac{1}{2\kappa} + n + 1)}}. \quad (17)$$

As a result, we get the normalized coherent state vectors

$$|z\rangle = \left[ \mathcal{N}_\kappa(|z|^2) \right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^{2n}}{\sqrt{(2\kappa)^{2n} n! \Gamma(\frac{1}{2\kappa} + n)}} \left[ |2n\rangle + \frac{z}{\sqrt{1 + 2\kappa n}} |2n + 1\rangle \right]. \quad (18)$$

The normalization factor  $\mathcal{N}_\kappa(|\psi|^2)$  is given by

$$\mathcal{N}_\kappa(|z|^2) = e_\kappa \left( \frac{|z|^2}{2\kappa} \right)$$

where the function  $e_\kappa(\cdot)$  is defined by

$$e_\kappa(x) = \sum_{n=0}^{\infty} \left( \frac{1}{2\kappa} + n + x \right) \frac{x^{2n}}{n! \Gamma(\frac{1}{2\kappa} + n + 1)}.$$

The coherent states provide a polynomial realization of the algebra  $\mathcal{B}_\kappa(1)$ . Indeed, Any state  $|\Psi\rangle$

$$|\Psi\rangle = \sum_{n=0}^{\infty} \left( \Psi_{2n} |2n\rangle + \Psi_{2n+1} |2n + 1\rangle \right)$$

is represented, in the Bargmann representation, as a function of the complex variable  $z$  according the following correspondence

$$|\Psi\rangle \longrightarrow \Psi(z) = \left[ \mathcal{N}_\kappa(|z|^2) \right]^{\frac{1}{2}} \langle \bar{z} | \Psi \rangle = \sum_{n=0}^{\infty} \left( \Psi_{2n} f_{2n} + \Psi_{2n+1} f_{2n+1} \right) \quad (19)$$

where the monomials  $f_{2n}(z)$  and  $f_{2n+1}(z)$  are

$$f_{2n}(z) = \left[ \mathcal{N}_\kappa(|z|^2) \right]^{\frac{1}{2}} \langle \bar{z} | 2n \rangle = \frac{z^{2n}}{\sqrt{(2\kappa)^{2n} n! \Gamma(\frac{1}{2\kappa} + n)}} \quad (20)$$

and

$$f_{2n+1}(z) = \left[ \mathcal{N}_\kappa(|z|^2) \right]^{\frac{1}{2}} \langle \bar{z} | 2n + 1 \rangle = \frac{z^{2n+1}}{\sqrt{(2\kappa)^{2n+1} n! \Gamma(\frac{1}{2\kappa} + n + 1)}} \quad (21)$$

correspond to the analytical representation of the vectors  $|2n\rangle$  and  $|2n + 1\rangle$  respectively. In this respect, the Fock space spanned by the number basis  $\{|n\rangle, n = 0, 1, 2, \dots\}$  is equivalent to the space of functions of the variable  $z$  spanned by the basis  $\{f_0(z), f_1(z), f_2(z), \dots\}$ . Furthermore, using the equations (18), (20) and (21), we realize the creation, annihilation and number operators, in the Bargmann representation, as follows

$$f^+ \equiv z \quad f^- \equiv \frac{1}{z} F_+ \left( z \frac{d}{dz} \right) \quad N \equiv z \frac{d}{dz}$$

where  $F$  is the structure function defined by (8) or equivalently by (11). Using the expressions (20) and (21), it is simple to check that

$$N f_{2n}(z) = 2n f_{2n}(z) \quad N f_{2n+1}(z) = (2n+1) f_{2n+1}(z).$$

The creation operator acts as multiplication by  $z$  and gives

$$f^+ f_{2n}(z) = \sqrt{1+2\kappa n} f_{2n+1}(z) \quad f^+ f_{2n+1}(z) = \sqrt{2\kappa(n+1)} f_{2n+2}(z).$$

To write the explicit form of the action of the annihilation operator, we note that the action of the operator  $(-)^N$  on any analytical function of the form (19) induces the parity transformation  $z \rightarrow -z$ . Then, using the result (11), the action of the annihilation operator  $f^-$  can be represented as

$$D_z^\kappa = \kappa \frac{d}{dz} + (1-\kappa) \frac{\partial}{\partial_{-1}z} \quad (22)$$

where the second object in the right-hand side is defined by

$$\frac{\partial}{\partial_{-1}z} = \frac{f(z) - f(-z)}{2z} \quad (23)$$

is a special form of the Fibonacci difference operator [16]. Using (22) together with (23), one obtains

$$f^- f_{2n}(z) = \sqrt{2\kappa n} f_{2n-1}(z) \quad f^- f_{2n+1}(z) = \sqrt{1+2\kappa n} f_{2n}(z).$$

reflecting that the differential operator  $D_z^\kappa$  acts as a generalized derivative in the Bargmann representation of the generalized fermion algebra  $\mathcal{B}_\kappa(1)$ . It is easy to check the anticommutation relation

$$\left\{ \frac{\partial}{\partial_{-1}z}, z \right\} = 1 \quad (24)$$

which is reminiscent of Grassmann variables. The generalized derivative  $D_z^\kappa$  is the sum of two derivatives: the usual(bosonic) one and the second is of Grassmann type. Finally, using the relation (24), one has

$$\{D_z^\kappa, z\} = 1 + 2\kappa z \frac{d}{dz}. \quad (25)$$

Now, we consider the special case  $\kappa = 0$ . It is important to emphasize that for  $\kappa \rightarrow 0$ , the coefficients  $c_{2n}(z)$  and  $c_{2n+1}(z)$ , given by (17), go to infinity for  $n > 0$ . Therefore, the terms in  $z^{2n}$  ( $n > 0$ ) in Eq. (14) make sense only if  $z$  is a Grassmann variable with

$$z^2 = 0.$$

Hence, the coherent state (14) reduces to

$$|z\rangle = \frac{1}{(1+z\bar{z})^{\frac{1}{2}}} [|0\rangle + z|1\rangle],$$

and the Fock basis corresponds

$$|0\rangle \rightarrow f_0(z) = 1 \quad |1\rangle \rightarrow f_1(z) = z.$$

In this limiting case, the generalized derivative  $D_z^\kappa$  (22) coincides with the partial derivative of the Grassmann variable  $z$  and satisfies the nilpotency condition

$$\left( \frac{\partial}{\partial_{-1}z} \right)^2 = 0$$

because the Bargmann space has a two dimensional basis  $\{1, z\}$ .



## 5 Normal ordering process

In quantum field theory a product of quantum fields, or equivalently their creation and annihilation operators, is usually said to be normal ordered (also called Wick order) when all creation operators are to the left of all annihilation operators in the product. The process of putting a product into normal order is called normal ordering (also called Wick ordering). The process of normal ordering is particularly important for a quantum mechanical Hamiltonian. When quantizing a classical Hamiltonian there is some freedom when choosing the operator order, and these choices lead to differences in the ground state energy. In this subsection we deal with the normal ordering process for the generalized fermion algebra  $\mathcal{B}_\kappa(1)$ . The normal ordering is obtained by using the Wick's theorem;

$$(f^+ f^-)^r = \sum_{k=1}^r (f^+)^k S_\kappa(r, k, N) (f^-)^k \quad (26)$$

where  $S_\kappa(r, k, N)$  is the  $\kappa$ -deformed Stirling operator of the second kind. Using the identity  $(f^+ f^-)^{n+1} = (f^+ f^-)(f^+ f^-)^n$ , we obtain the recurrence relation

$$\begin{aligned} S_\kappa(r+1, k, N) &= (-1)^{k-1} S_\kappa(r, k-1, N+1) \\ &+ (-1)^{k-1} \left( \frac{1-(-1)^k}{2} (1+2\kappa N) + \kappa k (-1)^k + \frac{1-(-1)^k}{2} \kappa \right) S_\kappa(r, k, N) \end{aligned} \quad (27)$$

The first few  $\kappa$ -deformed Stirling operator of the second kind are

$$S_\kappa(1, 1, N) = I, \quad (28)$$

$$S_\kappa(2, 1, N) = I + 2\kappa N, \quad S_\kappa(2, 2, N) = -I \quad (29)$$

$$S_\kappa(3, 1, N) = (I + 2\kappa N)^2, \quad S_\kappa(3, 2, N) = -I - 2\kappa N, \quad S_\kappa(3, 3, N) = I \quad (30)$$

$$S_\kappa(4, 1, N) = (I + 2\kappa N)^3, \quad S_\kappa(4, 2, N) = -(I - 2\kappa + 4\kappa^2) - 4\kappa(1 + \kappa)N - 4\kappa^2 N^2, \quad (31)$$

$$S_\kappa(4, 3, N) = -4\kappa, \quad S_\kappa(4, 4, N) = -I. \quad (32)$$

The  $\kappa$ -deformed Bell operator is then defined as

$$B_r(N) = \sum_{k=1}^r S_\kappa(r, k, N). \quad (33)$$

Some of the  $\kappa$ -deformed Bell operators are:

$$B_1(N) = I \quad (34)$$

$$B_2(N) = 2\kappa N \quad (35)$$

$$B_3(N) = I + 2\kappa N + 4\kappa^2 N^2 \quad (36)$$

$$B_4(N) = -(I + 2\kappa + 4\kappa^2) + 2\kappa(1 - 2\kappa)N + 8\kappa^2 N^2 + 8\kappa^3 N^3, \quad (37)$$

Finally, in the limit  $\kappa \rightarrow 0$ , using the equations (27) and (33), one gets

$$B_1(N) = I, \quad B_2(N) = 0, \quad (38)$$

and for  $r \geq 3$ , one shows

$$B_r(N) = \begin{cases} (-1)^r I & (r = 0 \pmod{3}) \\ (-1)^{r+1} I & (r = 1 \pmod{3}) \\ 0 & (r = 2 \pmod{3}) \end{cases} \quad (39)$$

which correspond to the Bell operators for ordinary fermion algebra.

## 6 Concluding remarks and perspectives

The main idea of this work is the "*fermionization*" of the generalized Weyl-Heisenberg algebra  $\mathcal{A}_\kappa(1)$  investigated in [11]. This "*fermionization*" procedure is introduced by simply replacing the commutator of creation and annihilation operators, in the structure relations of  $\mathcal{A}_\kappa(1)$ , by the anti-commutator. The resulting algebra  $\mathcal{B}_\kappa$  covers the ordinary fermion algebra. In contrast with the several extensions of fermionic algebra introduced in the context of quantum algebras [14], the algebra  $\mathcal{B}_\kappa(1)$  is not equivalent to the usual fermionic algebra. The parameter  $\kappa$  induces a drastic deviation from fermions to bosons. Indeed, the Fock and Bargmann realizations show clearly that the representation spaces is infinite dimensional and become suddenly two dimensional when  $\kappa$  approaches zero. This deviation also arises in the context of the coherent states constructed in this work. Indeed, we have shown that in the limiting situation  $\kappa \rightarrow 0$ , they reduce to fermionic coherent states involving Grassmann variables. In view of the connection between the algebra  $\mathcal{B}_\kappa(1)$  and the one-dimensional Calogero-Sutherland system presented in this work, it is worthwhile to look for the appropriate scheme to generalize  $\mathcal{B}_\kappa(1)$  to describe other potentials. Also, we notice that the representation of  $\mathcal{A}_\kappa(2)$  algebra [12] was generalized to incorporate multi-bosons [17]. Hilbertian as well as analytical representations were discussed. In this respect, it is interesting to investigate the multimode generalization of the generalized fermion algebra  $\mathcal{B}_\kappa(1)$ . Some preliminaries results, regarding this issue, were obtained [18] and we hope report on this subject in a forthcoming paper.

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